R A P P O R T D E R E C H E R C H E

# L R I

## LONG ALTERNATING CYCLES IN EDGE-COLORED COMPLETE GRAPHS

LI H / WANG G / ZHOU S

Unité Mixte de Recherche 8623
CNRS-Université Paris Sud – LRI

12/2007

**Rapport de Recherche N° 1481**

**CNRS – Université de Paris Sud**
Centre d'Orsay
LABORATOIRE DE RECHERCHE EN INFORMATIQUE
Bâtiment 490
91405 ORSAY Cedex (France)

# Long alternating cycles in edge-colored complete graphs

Hao Li[1,3], Guanghui Wang[1,2], Shan Zhou[1,3]

[1] Laboratoire de Recherche en Informatique, UMR 8623, C.N.R.S-Universite de Paris-sud, 91405-Orsay cedex, France
[2] School of Mathematics and System Science, Shandong University, 250100 Jinan, Shandong, China
[3] School of Mathematics and Statistics, Lanzhou University, 730000 Lanzhou, China

**Abstract.** Let $K_n^c$ denote a complete graph on $n$ vertices whose edges are colored in an arbitrary way. And let $\Delta(K_n^c)$ denote the maximum number of edges of the same color incident with a vertex of $K_n^c$. A properly colored cycle (path) in $K_n^c$, that is, a cycle (path) in which adjacent edges have distinct colors is called an alternating cycle (path). Our note is inspired by the following conjecture by B. bollobás and P. Erdős(1976): If $\Delta(K_n^c) < \lfloor n/2 \rfloor$, then $K_n^c$ contains an alternating Hamiltonian cycle. We prove that if $\Delta(K_n^c) < \lfloor n/2 \rfloor$, then $K_n^c$ contains an alternating cycle with length at least $\lceil \frac{n+2}{3} \rceil + 1$.

**Keywords:** alternating cycle, color degree, edge-colored graph

## 1 Introduction and notation

We use [2] for terminology and notations not defined here. Let $G = (V, E)$ be a graph. An *edge-coloring* of $G$ is a function $C : E \to N$($N$ is the set of nonnegative integers). If $G$ is assigned such a coloring $C$, then we say that $G$ is an *edge-colored graph*, or simply *colored graph*. Denote by $(G, C)$ the graph $G$ together with the coloring $C$ and by $C(e)$ the *color* of the edge $e \in E$. For a subgraph $H$ of $G$, let $C(H) = \{C(e) : e \in E(H)\}$ and $c(H) = |C(H)|$. For a color $i \in C(H)$, let $i_H = |\{e : C(e) = i \text{ and } e \in E(H)\}|$ and say that *color $i$ appears $i_H$ times in $H$*. For an edge-colored graph $G$, if $c(G) = c$, we call it a *$c$-edge colored* graph.

A properly colored cycle (path) in an edge-colored graph, that is, a cycle(path) in which adjacent edges have distinct colors is called an *alternating* cycle (path). In particular, an *alternating Hamiltonian* cycle (path) is a properly colored Hamiltonian cycle (path). For $l \geq 3$, let $AC_l$ denote an alternating cycle with length $l$. Besides a number of applications in graph theory and algorithms, the concept of alternating paths and cycles, appears in various other fields: genetics (cf. [9,10,11]), social sciences (cf. [8]). A good resource on alternating paths and cycles is the survey paper [2] by Bang-Jensen and Gutin.

Grossman and Häggkvist [12] were the first to study the problem of the existence of the alternating cycles in $c$-edge colored graphs. They proved Theorem 1.1 below in the case $c = 2$. The case $c \geq 3$ was proved by Yeo [17]. Let $v$ be a cut vertex in an edge-colored graph $G$. We say that $v$ *separates colors* if no component of $G - v$ is joined to $v$ by at least two edges of different colors.

**Theorem 1.1** (Grossman and Häggkvist [12], and Yeo [17]). *Let $G$ be a $c$-edge colored graph, $c \geq 2$, such that every vertex of $G$ is incident with at least two edges of different colors. Then either $G$ has a cut vertex separating colors, or $G$ has an alternating cycle.*

Given an edge-colored graph $G$, let $d^c(v)$, named the color degree of a vertex $v$, be defined as the maximum number of edges adjacent to $v$, that have distinct colors. In [16], some color degree conditions for the existence of alternating cycles are obtained as follows.

**Theorem 1.2** (Li and Wang [16]). *Let $G$ be an edge-colored graph with order $n \geq 3$. If $d^c(v) > \frac{n+1}{3}$ for every $v \in V(G)$, then $G$ has an alternating cycle $AC$ such that each color in $C(AC)$ appears at most two times in $AC$.*

**Theorem 1.3** (Li and Wang [16]). *Let $G$ be an edge-colored graph with order $n \geq 3$. If $d^c(v) \geq \frac{37n-17}{75}$ for every $v \in V(G)$, then $G$ contains at least one alternating triangle or one alternating quadrilateral.*

**Theorem 1.4** (Li and Wang [16]). *Let $G$ be an edge-colored graph with order $n$. If $d^c(v) \geq d \geq 2$, for every vertex $v \in V(G)$, then either $G$ has an alternating path with length at least $2d$, or $G$ has an alternating cycle with length at least $\lceil \frac{2d}{3} \rceil + 1$.*

Consider the edge-colored complete graph, we use the notation $K_n^c$ to denote a complete graph on $n$ vertices, each edge of which is colored by a color from the set $\{1, 2, \cdots, c\}$. And $\Delta(K_n^c)$ is the maximum number of edges of the same color adjacent to a vertex of $K_n^c$. And we have the following conjecture due to Bollobás and Erdős [4].

**Conjecture 1.5** (Bollobás and Erdős [4]). *If $\Delta(K_n^c) < \lfloor \frac{n}{2} \rfloor$, then $K_n^c$ contains an alternating Hamiltonian cycle.*

Bollobás and Erdős managed to prove that $\Delta(K_n^c) < \frac{n}{69}$ implies the existence of an alternating Hamiltonian cycle in $K_n^c$. This result was improved by Chen and Daykin [7] to $\Delta(K_n^c) < \frac{n}{17}$ and by Shearer [15] to $\Delta(K_n^c) < \frac{n}{7}$. So far the best asymptotic estimate was obtained by Alon and Gutin [1]

**Theorem 1.6** (Alon and Gutin [1]). *For every $\epsilon > 0$ there exists an $n_o = n_0(\epsilon)$ so that for every $n > n_o$, $K_n^c$ satisfying $\Delta(K_n^c) \leq (1 - \frac{1}{\sqrt{2}} - \epsilon)n$ has an alternating Hamiltonian cycle.*

In the present paper, we study the long alternating cycles of edge-colored complete graphs and gain the following result.

**Theorem 1.7** *If $\Delta(K_n^c) < \lfloor \frac{n}{2} \rfloor$, then $K_n^c$ contains an alternating cycle with length at least $\lceil \frac{n+2}{3} \rceil + 1$.*

## 2 Proof of Theorem 1.7

If $P = v_1 v_2 \cdots v_p$ is a path, let $P[v_i, v_j]$ denote the subpath $v_i v_{i+1} \cdots v_j$, and $P^-[v_i, v_j] = v_j v_{j-1} \cdots v_i$.

**Lemma 2.1** (Bang-Jensen, Gutin and Yeo[3]). *If $K_n^c$ contains a properly colored 2-factor, then it has a properly colored Hamiltonian path.*

Häggkvist [13] announced a non-trivial proof of the fact that every edge-colored complete graph graph satisfying above Bollobás-Erdős condition contains a properly colored 2-factor. Lemma 2.1 and Häggkvist's result imply that every edge-colored complete graph satisfying Bollobás-Erdős condition has an alternating Hamiltonian path.

*Proof of Theorem 1.7.*

If $n = 3$, the conclusion holds clearly. So we assume that $n \geq 4$. By contradiction. Suppose that our conclusion does not hold. Then let $P = v_1 v_2 \cdots v_n$ be an alternating Hamiltonian path of $K_n^c$. Choose $s$ satisfying the followings:

$R_1$. $C(v_1 v_s) \neq C(v_1 v_2)$.
$R_2$. $s \geq \lceil \frac{n+2}{3} \rceil + 1$.
$R_3$. Subject to $R_1, R_2$, $s$ is minimum.

**Lemma 2.2**
(1.1) $s \leq \lfloor \frac{n}{2} \rfloor + \lceil \frac{n+2}{3} \rceil - 1$.
(1.2) *For $i \geq s$, if $C(v_1 v_i) \neq C(v_1 v_2)$, then $C(v_1 v_i) \neq C(v_i v_{i+1})$.*

**Proof.** By $R_3$, for $\lceil \frac{n+2}{3} \rceil + 1 \leq j \leq s - 1$, we have that $C(v_1 v_j) = C(v_1 v_2)$. If $s \geq \lfloor \frac{n}{2} \rfloor + \lceil \frac{n+2}{3} \rceil$, then there exist at least $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n+2}{3} \rceil - (1 + \lceil \frac{n+2}{3} \rceil) + 1 \geq \lfloor \frac{n}{2} \rfloor$ edges with the color $C(v_1 v_2)$ incident with $v_1$, a contradiction with $\Delta(K_n^c) < \lfloor \frac{n}{2} \rfloor$.

Since $P$ is an alternating Hamiltonian path, then $C(v_{i-1} v_i) \neq C(v_i v_{i+1})$. If there exists $i \geq s$ such that $C(v_1 v_i) \neq C(v_1 v_2)$ and $C(v_1 v_i) = C(v_i v_{i+1})$, then $P[v_1, v_i] v_i v_1$ is an alternating cycle with length $i \geq s \geq \lceil \frac{n+2}{3} \rceil + 1$, a contradiction.

Then choose $t$ satisfying the followings:

$R_1^{'}$. $C(v_t v_n) \neq C(v_{n-1} v_n)$.

$R_2'$. $t \le n - \lceil \frac{n+2}{3} \rceil$.
$R_3'$. Subject to $R_1', R_2'$, $t$ is maximum.

Similarly, we have the following lemma, here we omit the details.

**Lemma 2.3**
(3.1) $t \ge \lceil \frac{n}{2} \rceil - \lceil \frac{n+2}{3} \rceil + 2$.
(3.2) *For* $i \le t$, *if* $C(v_i v_n) \ne C(v_{n-1} v_n)$, *then* $C(v_i v_n) \ne C(v_{i-1} v_i)$.

**Lemma 2.4.** $s < t$.

**Proof.** Otherwise, we have that $s \ge t$. If $s > t$, then $AC^0 = v_1 v_s P[v_s, v_n] v_l v_t P^-[v_t, v_1]$ is an alternating cycle. And $|AC^0| = |P[v_s, v_n]| + |P[v_1, v_t]| \ge (n - \lfloor \frac{n}{2} \rfloor - \lceil \frac{n+2}{3} \rceil + 2) + (\lceil \frac{n}{2} \rceil - \lceil \frac{n+2}{3} \rceil + 2) = 2(\lceil \frac{n}{2} \rceil - \lceil \frac{n+2}{3} \rceil) + 4 \ge \lceil \frac{n+2}{3} \rceil + 1$, a contradiction.

So we assume that $s = t$. For $s+1 \le j \le n-1$, we conclude that $C(v_1 v_j) = C(v_1 v_2)$. Otherwise, there is an alternating cycle $AC^1 = v_1 v_j P[v_j, v_n] v_n v_s P^-[v_s, v_n]$ with length $|AC^1| \ge 2 + |V(P[v_1, v_s])| \ge 3 + \lceil \frac{n+2}{3} \rceil$, which gives a contradiction. Similarly, for $2 \le j \le s-1$, it holds that $C(v_j v_n) = C(v_{n-1} v_n)$. Then by $\Delta(K_n^c) < \lfloor \frac{n}{2} \rfloor$, consider vertex $v_1$ and the color $C(v_1 v_2)$, it holds that $n - s < \lfloor \frac{n}{2} \rfloor$, then $s > \lceil \frac{n}{2} \rceil$. Similarly, consider vertex $v_n$ and the color $C(v_{n-1} v_n)$, we have that $s - 1 < \lfloor \frac{n}{2} \rfloor$, then $s < \lfloor \frac{n}{2} \rfloor + 1$, a contradiction.

**Lemma 2.5.** *For* $2 \le j \le s-1$, $C(v_n v_j) = C(v_{n-1} v_n)$; *And for* $t+1 \le j \le n-1$, $C(v_1 v_j) = C(v_1 v_2)$.

**Proof.** By symmetry, we only prove the first part. Otherwise, there exists $2 \le j \le s-1$ such that $C(v_j v_n) \ne C(v_{n-1} v_n)$. Clearly, $j \le t$, thus by Lemma 2.3 we have that $C(v_{j-1} v_j) \ne C(v_j v_n)$. Then we get an alternating cycle $AC^2 = v_1 v_s P[v_s, v_n] v_n v_j P^-[v_j, v_1]$. And it holds that $|AC^2| \ge |V(P[v_s, v_n])| + 2 \ge |V(P[v_t, v_n])| + 3 \ge \lceil \frac{n+2}{3} \rceil + 3$, a contradiction.

Denote $A = \{v : C(v_1 v) \ne C(v_1 v_2)\}$ and $B = \{v : C(v_n v) \ne C(v_{n-1} v_n)\}$.

**Lemma 2.6.** $|A \cap V(P[v_s, v_t])| + |B \cap V(P[v_s, v_t])| \ge 2(\lceil \frac{n}{2} \rceil - \lceil \frac{n+2}{3} \rceil + 1)$.

**Proof.** By $R_1$, $|A \cap V(P[v_s, v_n])| \ge n - (\lfloor \frac{n}{2} \rfloor - 1) - (\lceil \frac{n+2}{3} \rceil - 1) \ge \lceil \frac{n}{2} \rceil - \lceil \frac{n+2}{3} \rceil + 2$. Then by Lemma 2.5, we obtain that $A \cap V(P[v_s, v_n]) = A \cap (V(P[v_s, v_t]) \cup \{v_n\}) = (A \cap V(P[v_s, v_t])) \cup (A \cap \{v_n\})$. It follows that $|A \cap V(P[v_s, v_t])| \ge \lceil \frac{n}{2} \rceil - \lceil \frac{n+2}{3} \rceil + 1$. Similarly, we can obtain that $|B \cap V(P[v_s, v_t])| \ge \lceil \frac{n}{2} \rceil - \lceil \frac{n+2}{3} \rceil + 1$. Then $|A \cap V(P[v_s, v_t])| + |B \cap V(P[v_s, v_t]| \ge 2(\lceil \frac{n}{2} \rceil - \lceil \frac{n+2}{3} \rceil + 1)$.

Now we completes the proof of Theorem 1.7 as follows. We have that $|V(P[v_s, v_t])| \le n - |V(P[v_1, v_{s-1}])| - |V(P[v_{t+1}, v_l])| \le n - 2\lceil \frac{n+2}{3} \rceil$. And by Lemma 2.6, $|A \cap V(P[v_s, v_t])| + |B \cap V(P[v_s, v_t])| = |A| + |B| \ge 2(\lceil \frac{n}{2} \rceil - \lceil \frac{n+2}{3} \rceil + 1) > n - 2\lceil \frac{n+2}{3} \rceil + 1 > |V(P[v_s, v_t])|$, then it follows that there exists $v_j$ $(s+1 \le j \le t)$ such that $v_j \in A$ and $v_{j-1} \in B$. So we get an alternating Hamiltonian cycle $v_1 v_j P[v_j, v_n] v_n v_{j-1} P^-[v_{j-1}, v_1]$, a contradiction. This completes the proof.

## References

1. N. Alon and G. Gutin, Properly colored Hamiltonian cycles in edge colored complete graphs, Random Structures and Algorithms 11(1997), 179-186
2. J. Bang-Jensen and G. Gutin, Alternating cycles and paths in edge-colored multigraphs: a survey, Discrete Math. 165-166(1997), 39-60.
3. J. Bang-Jensen, G. Gutin and A. Yeo, Properly coloured Hamiltonian paths in edge-colored complete graphs, Discrete Appl. Math. 82(1998), 247-250.
4. B. Bollob*á*s and P. Erdős, Alternating Hamiltonian cycles, Israel J. Math. 23(1976), 126-131.
5. J.A. Bondy and U.S.R. Murty. Graph Theory with Applications, Macmillan Press[M]. New York, 1976.
6. L. Caccetta and R. Häggkvist, On minimal digraphs with given girth, in "Proceedings, Ninth S-E Conference on Combinatorics, Graph Theory and Computing, 1978", 181-187.
7. C.C. Chen ands D.E. Daykin, Graphs with Hamiltonian cycles having adjacent lines different colors. J. Combin. Theory Ser B21(1976), 135-139.
8. W.S. Chow, Y. Manoussakis, O. Megalakaki, M. Spyratos and Z. Tuza, Paths through fixed vertices in edge-colored graphs, J. des Mathematqiues, Informatique et Science, Humaines 32(1994), 49-58
9. D. Dorninger, On permutations of chromosomes, in: Contributions to General Algebra, vol. 5(Verlag Hölder-Pichler-Tempsky, Wien; Teubner, Stuttgart, 1987), 95-103.
10. D. Dorninger, Hamiltonian circuits determing the order of chromosomes, Discrete Appl. Math. 50(1994), 159-168.
11. D. Dorninger and W. Timischl, Geometrical constraints on Bennett's predictions of chromosome order. Heredity 58(1987), 321-325.
12. J.W. Grossman and R. H*ä*ggkvist, Alternating cycles in edge-partitioned graphs. J. Combin. Theory Ser B 34(1983),77-81
13. R. Häggkvist, A talk at Intern. Colloquium on Combinatorics and Graph Theory at Balatonlelle, Hungary, July 15-19, 1996
14. G. Hahn and C. Thomassen, Path and cycle sub-Ramsey numbers and edge-coloring conjecture, Discrete Math. 62(1)(1986), 29-33.
15. J. Shearer, A property of the colored complete graph. Discrete Math. 25(1979), 175-178.
16. H. Li and G. Wang, Color degree and alternating cycles in edge-colored graphs, RR L.R.I NO1461(2006)
17. A. Yeo, Alternating cycles in edge-coloured graphs. J. Combin. Theory Ser B 69(1997), 222-225.