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The k-dominating cycles in graphs*

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Abstract

For a graph G, let $\bar{\sigma}_{k+3}(G) = \min\ \{d(x_1) + d(x_2) + \cdots + d(x_{k+3}) - |N(x_1) \cap N(x_2) \cap \cdots \cap N(x_{k+3})| \mid x_1, x_2, \cdots, x_{k+3}$ are $k+3$ independent vertices in $G\}$. In [5], H. Li proved that if G is a 3-connected graph of order n and $\bar{\sigma}_4(G) \geq n+3$, then G has a maximum cycle such that each component of $G-C$ has at most one vertex. In this paper, we extend this result as follows. Let G be a $(k+2)$-connected graph of order n. If $\bar{\sigma}_{k+3}(G) \geq n + k(k+2)$, G has a cycle C such that each component of $G-C$ has at most k vertices. Moreover, the lower bound is sharp.

Keywords: cycle, neighborhood, degree sum, k-dominating

1 Introduction and Notations

All the graphs considered in this paper are undirected and simple. We use [1] for terminology and notations not defined here. Let $C = c_1c_2...c_pc_1$ be a cycle in graph G. We use $C[c_i, c_j]$ to denote the sub-path $c_ic_{i+1}...c_j$, and $\bar{C}[c_j, c_i]$ to denote the sub-path $c_jc_{j-1}...c_i$, where the indices are taken modulo p. We will consider $C[c_i, c_j]$ and $\bar{C}[c_j, c_i]$ both as paths and as vertex sets. Define $C(c_i, c_j] = C[c_{i+1}, c_j]$, $C[c_i, c_j) = C[c_i, c_{j-1}]$ and $C(c_i, c_j) = C[c_{i+1}, c_{j-1}]$. We use similar definitions for a path. We give C a fixed orientation. For any i, we put $c_i^+ = c_{i+1}$, $c_i^- = c_{i-1}$, $c_i^{+2} = c_{i+2}$ and $c_i^{-2} = c_{i-2}$. For a vertex set $A \subseteq C$, $A^+ = \{v^+ \mid v \in A\}$, $A^- = \{v^- \mid v \in A\}$, $A^{+2} = (A^+)^+$ and $A^{-2} = (A^-)^-$. For a vertex x of G, a neighbor of x means a vertex adjacent to x, denoted by $N_G(x)$, and the degree of x is the number of neighbors of x, denoted by $d(x)$. Let $N_C^-(x) = \{c_i \mid c_i^+ \in N_C(x)\}$ and $N_C^{-2}(x) = \{c_i \mid c_i^{+2} \in N_C(x)\}$. A maximal connected subgraph of G is called a *component* of G. Let $R = G - C$ be the induced subgraph in G by $V(G) - V(C)$. Denote by

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$R(C[c_i, c_j])$ the induced subgraph in G by the union of the components in G that is adjacent to some vertex in $C[c_i, c_j]$ and $R^*(C[c_i, c_j]) = R(C[c_i, c_j]) \cup C[c_i, c_j]$. Define $\bar{\sigma}_k(G) = \min\ \{d(x_1) + d(x_2) + \cdots + d(x_k) - |N(x_1) \cap N(x_2) \cap \cdots \cap N(x_k)| \mid x_1, x_2, \cdots, x_k$ are k independent vertices in $G\}$ and $\sigma_k(G) = \min\ \{d(x_1) + d(x_2) + \cdots + d(x_k) \mid x_1, x_2, \cdots, x_k$ are k independent vertices in $G\}$. A graph G is called to be *hamiltonian* if there is a cycle that contains all vertices of G. A cycle C is called *k-dominating* if no component of $G - C$ has more than k vertices. Clearly, a *hamiltonian* cycle is a 0-dominating cycle and a 1-dominating cycle is called *dominating* cycle.

Various long cycle problems are interesting and important in graph theory and have been deeply studied. Two classical results are due to Dirac and Ore respectively.

Theorem 1.1 (Dirac [3]) *Let G be a graph on $n \geq 3$ vertices. If the minimum degree $\delta(G) \geq \frac{n}{2}$, G is hamiltonian.*

Theorem 1.2 (Ore [8]) *Let G be a graph on $n \geq 3$ vertices. If $\sigma_2(G) \geq n$, G is hamiltonian.*

It is natural to consider sufficient conditions concerning the degree sum of more independent vertices. Flandrin, Jung and Li [4] investigated the degree sum of three independent vertices and obtained the following result.

Theorem 1.3 (Flandrin, Jung and Li [4]) *Let G be a 2-connected graph of order n. If $\bar{\sigma}_3(G) \geq n$, G is hamiltonian.*

Based on the reason that it is too difficult to obtain the sufficient conditions for a graph to be hamiltonian by considering the degree sum of four or more independent vertices, many authors turn into investigating the sufficient conditions for a graph to have a dominating cycle and the relation between dominating cycle and the longest cycle concerning the degree sum of independent vertices. In [7], Nash-Williams gave a sufficient condition for each longest cycle of a 2-connected graph to be a dominating cycle.

Theorem 1.4 (Nash-Williams [7]) *Let G be a 2-connceted graph on n vertices with $\delta(G) \geq \frac{n+2}{3}$. Then every longest cycle in G is a dominating cycle.*

Bondy [2] generalized this result to the degree sum of three indpendent vertices.

Theorem 1.5 (Bondy [2]) *Let G be a 2-connected graph of order $n \geq 3$ with $\sigma_3(G) \geq n + 2$. Then each longest cycle of G is a dominating cycle.*

Futher, Lu et al. [6] proved the following result.

Theorem 1.6 (Lu et al. [6]) *Let G be a 3-connected graph of order $n \geq 13$. If $\sigma_4(G) \geq \frac{4}{3}n + \frac{5}{3}$, then each longest cycle of G is a dominating cycle.*

H. Li [5] studied the degree sum of four independent vertices in 3-connected graphs and proved:

Theorem 1.7 (Li [5]) *Let G be a 3-connected graph of order n. If $\bar{\sigma}_4(G) \geq n + 3$, G has a dominating maximum cycle.*

In this paper, we extend this result to the degree sum of $k+3$ independent vertices and present the following result:

Theorem 1.8 *Let G be a (k+2)-connected graph of order n. If $\bar{\sigma}_{k+3}(G) \geq n + k(k+2)$, G has a cycle C such that each component of $G - C$ has at most k vertices.*

It can be seen that Theorem 1.3 and Theorem 1.7 are consistently with Theorem 1.8 when $k = 0$ and $k = 1$, respectively.

Theorem 1.8 is best possible as shown by the following example (see Fig. 1). The graph G is obtained by $k+3$ complete graphs K_{k+1} and $k+2$ vertices $v_1, v_2, ..., v_{k+2}$ by adding edges between v_i and each vertex in $k+3$ complete graphs K_{k+1}, $i = 1, 2, ..., k+2$, all of which are disjoint. We take a vertex u_i $(i = 1, 2, ..., k+3)$ from each of the $k+3$ copies of K_{k+1}. Then the $k+3$ vertices $u_1, u_2, ..., u_{k+3}$ are independent and

$$\begin{aligned} \bar{\sigma}_{k+3}(G) &= \sum_{i=1}^{k+3} d(u_i) - |\cap_{i=1}^{k+3} N(u_i)| \\ &= (k+3)(2k+2) - (k+2) = 2k^2 + 7k + 4 \\ &= (k^2 + 5k + 5) - 1 + k^2 + 2k = n - 1 + k(k+2). \end{aligned}$$

However, for each cycle C in G, there exists a component with $k+1$ vertices in $G - C$.

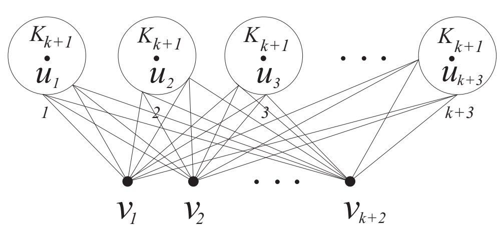

Figure 1.

The proof of Theorem 1.8 will be given in the next section.

2 Proof of Theorem 1.8

Suppose, to the contrary, that for each cycle C of G, there exists at least one component H of $G-C$ with $|H| \geq k+1$. We choose a cycle C such that:

(a) the number of component H^* in $G-C$ with $|H^*| \geq k+1$ is as small as possible.

(b) subject to (a), the component H in $G-C$ with $|H| \geq k+1$ is as small as possible.

We give C a fixed orientation. Since G is $(k+2)$-connected, H contains a vertex x_0 that has $t(\geq k+2)$ paths $P_1[x_0,v_1], P_2[x_0,v_2], ..., P_t[x_0,v_t]$ from x_0 to C having only x_0 in common. For any i, let $V(P_i)\cap V(C)=\{v_i\}$, and $v_1,v_2,...,v_t$ occur in this order along C with the chosen orientation. Denote $C_i=C(v_i,v_{i+1}]$, $i=1,2,...,t$. A vertex u of a segment C_i is said to be insertible, if there is an edge $xy \subseteq E(C(v_{i+1},v_i))$ such that ux and uy belong to $E(G)$. By the choice of C, for each $i\in\{1,2,...,k+2\}$, let x_i be the first non-insertible vertex in C_i and denote $F_i=C(x_i,v_{i+1}]$, $i=1,2,...,k+2$, where the indices are taken modulo t.

Remark 1. [5] $x_0,x_1,...,x_{k+2}$ are independent vertices.

Remark 2. [5] $R^*(N^-(x_i))\cap N(x_j)=\emptyset$, $1\leq i<j\leq k+2$.

Remark 3. [5] $N(x_i)\cap(\cup_{j=1}^{t}P_j(x_0,v_j))=\emptyset$, $i=1,2,...,k+2$.

Remark 4. [5] $N(x_i)\cap(\cup_{j\neq i}C(v_j,x_j))=\emptyset$, $i,j=1,2,...,k+2$.

Thus $\sum_{i=1}^{k+2} d_{C(v_j,x_j)}(x_i) \leq |C(v_j,x_j)|$, $j=1,2,...,k+2$. For each segment F_j, we use $P_{F_j}[x_i,y_i^k]$ to denote the kth path, that is, internally disjoint from C, from x_i to F_j, $i,j\in\{1,2,...,k+2\}$. Let $R^*(F_j(y_p^m,y_q^n))$ $(q<p)$ be a segment such that $(y_q^n)^{-h}=y_p^m$, $h\geq 2$ and $R^*(F_j(y_p^m,y_q^n))\cap(\cup_{i=1}^{k+2}N(x_i))=\emptyset$. We have the following claim.

Claim 1. $|R^*(F_j(y_p^m,y_q^n))|\geq k+1$, $\forall p,q\in\{1,2,...,k+2\}$.

Proof. We take a cycle $C'=x_0P_p(x_0,v_p)v_p\bar{C}(v_p,y_q^n)y_q^n\bar{P}_{F_j}(y_q^n,x_q)x_qC(x_q,y_p^m)y_p^m\bar{P}_{F_j}(y_p^m,x_p)$ $x_pC(x_p,v_q)v_qP_q(v_q,x_0)x_0$. By inserting the vertices of $C(v_p,x_p)$ and $C(v_q,x_q)$ into the corresponding inserting segments, we get a cycle with $H'=H-\{x_0\}$. By the choice of C, $|R^*(F_j(y_p^m,y_q^n))|\geq k+1$. □

Suppose that $R^*(F_i(y_p^k,y_q^l))$ $(q<p, i\leq j)$ is another different segment such that $(y_q^l)^{-r}=y_p^k$, $r\geq 2$ and $R^*(F_i(y_p^k,y_q^l))\cap(\cup_{i=1}^{k+2}N(x_i))=\emptyset$. If $R^*(F_i(y_p^k,y_q^l))\cap R^*(F_j(y_p^m,y_q^n))\neq\emptyset$, there are paths from $F_i(y_p^k,y_q^l)$ to $F_j(y_p^m,y_q^n)$ internally disjoint from $F_i(y_p^k,y_q^l)\cup F_j(y_p^m,y_q^n)$. We choose the last path zPz', in the sense that $R^*(F_i(z,y_q^l))\cap R^*(F_j(y_p^m,z'))=\emptyset$, where $z\in F_i(y_p^k,y_q^l)$ and $z'\in F_j(y_p^m,y_q^n)$. Take cycle $C'=x_0P_q(x_0,v_q)v_q\bar{C}(v_q,x_p)x_pP_{F_j}(x_p,y_p^m)y_p^m$ $\bar{C}(y_p^m,y_q^l)y_q^l\bar{P}_{F_i}(y_q^l,x_q)x_qC(x_q,z)zPz'C(z',v_p)v_p\bar{P}_p(v_p,x_0)x_0$(see the bold lines in Fig. 2). By inserting the vertices of $C(v_q,x_q)$ and $C(v_p,x_p)$ into the corresponding inserting segments, we get a new cycle with $H'=H-\{x_0\}$. By the choice of C, $|R^*(F_i(z,y_q^l))|\geq k+1$ or $|R^*(F_j(y_p^m,z'))|\geq k+1$. Assume that $|R^*(F_j(y_p^m,z'))|\geq k+1$. Redefine $R^*(F_j(y_p^m,y_q^n))=R^*(F_j(y_p^m,z'))$. Then $R^*(F_i(y_p^k,y_q^l))\cap R^*(F_j(y_p^m,y_q^n))=\emptyset$ and each has

at least $k+1$ vertices.

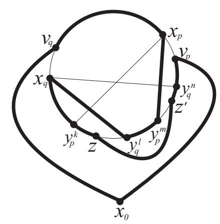

Figure 2.

Now, we consider the relation between $R^*(F_j(y_p, y_q))$ and other segment that is made by a pair different from x_p and x_q. Without cause of confusion, we breviate $R^*(F_j(y_p, y_q)) = L^0_{j_{pq}}$ $\forall p, q \in \{1, 2, ..., k+2\}$.

Let $L^0_{j_{mn}}$ and $L^0_{j_{pq}}$ be two different intersecting segments. Without loss of generality, assume that $x_p \neq x_n, x_m$. Since they are the segments in F_j, either $C(y_n, y_p) \cap \{x_n, x_m, x_q, x_p\} = \emptyset$ or $C(y_q, y_m) \cap \{x_n, x_m, x_q, x_p\} = \emptyset$. By symmetry, assume that $C(y_n, y_p) \cap \{x_n, x_m, x_q, x_p\} = \emptyset$. As $L^0_{j_{mn}} \cap L^0_{j_{pq}} \neq \emptyset$, similarly as above, we choose the last path zPz' from $F_j(y_m, y_n)$ to $F_j(y_p, y_q)$, where $z \in F_j(y_m, y_n)$ and $z' \in F_j(y_p, y_q)$. Take $C' = x_0 P_n(x_0, v_n) v_n \bar{C}(v_n, x_p) x_p P_{F_j}(x_p, y_p) y_p \bar{C}(y_p, y_n) y_n \bar{P}_{F_j}(y_n, x_n) x_n C(x_n, z) z P z' \bar{C}(z', v_p)$ $v_p \bar{P}_p(v_p, x_0) x_0$(see the bold lines in Fig. 3). By inserting the vertices of $C(v_n, x_n)$ and $C(v_p, x_p)$ into the corresponding inserting segments, we get a new cycle with $H' = H - \{x_0\}$. By the choice of C, $|R^*(F_j(z, y_n))| \geq k+1$ or $|R^*(F_j(y_p, z'))| \geq k+1$. Assume that $|R^*(F_j(y_p, z'))| \geq k+1$. Define $L^1_{j_{pq}} = R^*(F_j(y_p, z'))$ and $L^1_{j_{mn}} = L^0_{j_{mn}}$, $\forall m, n \in \{1, 2, ..., k+2\}$ and $\{m, n\} \neq \{p, q\}$. By repeating this process, we obtain a sequence of segments $L^0_{j_{pq}} \subseteq L^1_{j_{pq}} \subseteq ... \subseteq L^t_{j_{pq}}$ such that $L^t_{j_{pq}} \cap L^t_{j_{mn}} = \emptyset$, $\forall p, q, m, n \in \{1, 2, ..., k+2\}$ and $|L^t_{j_{pq}}| \geq k+1$.

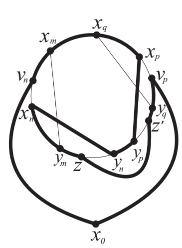

Figure 3.

Let $L^t_{i_{mn}}$ and $L^t_{j_{pq}}$ $(i < j)$ be two intersecting segments. By symmetry, we only consider the case that $|C(y_n, y_p) \cap \{x_n, x_m, x_q, x_p\}| \leq 2$. If $C(y_n, y_p) \cap \{x_n, x_m, x_q, x_p\} = \emptyset$, we can get two non-intersecting segments similarly as above and each has at least $k+1$ vertices. So assume that $C(y_n, y_p) \cap \{x_n, x_m, x_q, x_p\} \neq \emptyset$. Without loss of generality, assume that $x_m \in C(y_n, y_p)$. Similarly, we choose the last path $zPz^{'}$ from $F_i(y_m, y_n)$ to $F_j(y_p, y_q)$, where $z \in F_i(y_m, y_n)$ and $z^{'} \in F_j(y_p, y_q)$. If $x_n \notin C(y_n, y_p)$, take $C^{'} = x_0P_n(x_0, v_n)v_n\bar{C}(v_n, x_p)x_pP_{F_j}(x_p, y_p)y_p\bar{C}(y_p, y_n)y_n\bar{P}_{F_i}(y_n, x_n)x_nC(x_n, z)zPz^{'}C(z^{'}, v_p)v_p\bar{P}_p(v_p, x_0)x_0$ (see the bold lines in Fig. 4 (a)). If $x_n \in C(y_n, y_p)$, take $C^{'} = x_0P_n(x_0, v_n)v_n\bar{C}(v_n, y_n)y_n\bar{P}_{F_i}(y_n, x_n)x_nC(x_n, y_p)y_p\bar{P}_{F_j}(y_p, x_p)x_pC(x_p, z)zPz^{'}C(z^{'}, v_p)v_p\bar{P}_p(v_p, x_0)x_0$ (see the bold lines in Fig. 4 (b)). By inserting the vertices of $C(v_n, x_n)$ and $C(v_p, x_p)$ into the corresponding inserting segments, we get a new cycle with $H^{'} = H - \{x_0\}$. By the choice of C, $|R^*(F_i(z, y_n))| \geq k+1$ or $|R^*(F_j(y_p, z^{'}))| \geq k+1$ holds. Without loss of generality, suppose that $|R^*(F_j(y_p, z^{'}))| \geq k+1$. Define $L^{t+1}_{jpq} = R^*(F_j(y_p, z^{'}))$, $L^{t+1}_{imn} = L^t_{imn}$, $\forall m, n \in \{1, 2, ..., k+2\}$ and $\{m, n\} \neq \{p, q\}$. By continuing this process, for each $j \in \{1, 2, ..., k+2\}$, we obtain a sequence of segments $L^t_{jpq} \subseteq L^{t+1}_{jpq} \subseteq ... \subseteq L^s_{jpq}$ and $L^s_{imn} \cap L^s_{jpq} = \emptyset$, $\forall p, q, m, n \in \{1, 2, ..., k+2\}$.

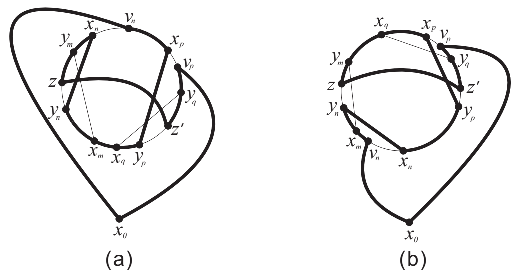

Figure 4.

For any $t \geq r > k+2$, let w_r be the first vertex in $C(v_r, v_{r+1}]$ such that $|R^*(C(v_r, w_r))| \geq k+1$.

Suppose that there exists a segment $L^s_{jpq} \cap R^*(C(v_r, w_r)) \neq \emptyset$. Let $zPz^{'}$ be the last path from $C(v_r, w_r)$ to $F_j(y_p, y_q)$, in the sense that $R^*(F_j(z^{'}, y_q)) \cap R^*(C(v_r, w_r)) = \emptyset$. Take $C^{'} = x_0P_q(x_0, v_q)v_q\bar{C}(v_q, z)zPz'\bar{C}(z', x_q)x_qP_{F_j}(x_q, y_q)y_qC(y_q, v_r)v_rP_r(v_r, x_0)x_0$(see the bold lines of Fig. 5 (a)). Let $zP'z^{'}$ be the first path from $C(v_r, w_r)$ to $F_j(y_p, y_q)$, in the sense that $R^*(C(v_r, w_r)) \cap R^*(F_j(y_p, z^{'})) = \emptyset$, where $z^{'} \in F_j(y_p, y_q)$ and $z \in C(v_r, w_r)$. Take $C^{'} = x_0P_p(x_0, v_p)v_p\bar{C}(v_p, z^{'})z^{'}\bar{P}^{'}zC(z, y_p)y_p\bar{P}_{F_j}(y_p, x_p)x_pC(x_p, v_r)v_r\bar{P}_r(v_r, x_0)x_0$(see the bold lines of Fig. 5 (b)). By inserting the vertices of $C(v_q, x_q)$ or $C(v_p, x_p)$ into the corresponding inserting segments, we get a new cycle with $H^{'} = H - \{x_0\}$. By the choice of w_r, $|R^*(C(v_r, z))| \leq k$ and then $|R^*(F_j(z^{'}, y_q))| \geq k+1$, or $|R^*(F_j(y_p, z^{'}))| \geq k+1$. Without loss of generality, assume that $|R^*(F_j(y_p, z^{'}))| \geq k+1$. Define $L^{s+1}_{jpq} = R^*(F_j(y_p, z^{'}))$. For each $i \in \{1, 2, ..., k+2\}$ and $\{m, n\} \neq \{p, q\}$, $L^{s+1}_{imn} = L^s_{imn}$.

Finally, we obtain a sequence of segments $L^0_{jpq} \subseteq ... \subseteq L^t_{jpq} \subseteq ... \subseteq L^s_{jpq} \subseteq ... \subseteq L^h_{jpq}$. By the above arguments, for each $i \neq j \in \{1, 2, ..., k+2\}$ and $p, q, m, n \in \{1, 2, ..., k+2\}$, the following claim holds.

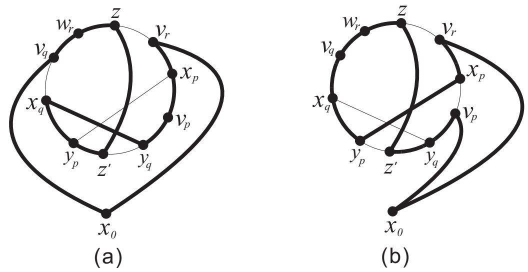

Figure 5.

Claim 2. *(1)* $|L^h_{j_{pq}}| \geq k+1$,

(2) $L^h_{j_{mn}} \cap L^h_{j_{pq}} = \emptyset$,

(3) $L^h_{i_{mn}} \cap L^h_{j_{pq}} = \emptyset$,

(4) $L^h_{j_{pq}} \cap R^*(C(v_r, w_r)) = \emptyset$, $r > k+2$.

Lemma 2.1 *[5] (1) There is no path between* x_0 *and a vertex in* $L^h_{j_{pq}}$ *with all internal vertices in* $G - C - P_j[x_0, v_j)$, *for any* $p, q \in \{1, 2, ..., k+2\}$ *and* $j = 1, 2, ..., t$,

(2) $R^*(C(v_r, w_r)) \cap N(x_i) = \emptyset$, *for* $1 \leq i \leq k+2$, $r > k+2$ *and*

(3) $R^*(C(v_r, w_r)) \cap R^*(C(v_{r'}, w_{r'})) = \emptyset$ *with* $r \neq r'$, $r, r' > k+2$.

For each $j \in \{1, 2, ..., k+2\}$, define $L^*_j = \cup_{p,q \in \{1,2,...,k+2\}} L^h_{j_{pq}}$ and $L_j = L^*_j \backslash F_j$. Then either $L^*_j = \emptyset$ or $|L^*_j| \geq k+1$. Now, for each $j \in \{1, 2, ..., k+2\}$, we regard the segment F_j as a path $P = v_1 v_2 ... v_p$ and compute the degree sum of $x_1, x_2, ..., x_{k+2}$ in $P \cup L_j$.

Lemma 2.2 *Let* G *be a simple graph,* $P = v_1 v_2 ... v_p$ *a path in* G *and* $x_1, x_2, ..., x_{k+2}$ *are* $k+2$ *vertices in* $V(G) - V(P)$ *such that* $N^-_P(x_i) \cap N_P(x_j) = \emptyset$, $1 \leq i < j \leq k+2$, *and* $N^-_P(x_i) \cap N_P(x_i) = \emptyset$, $1 \leq i \leq k+2$. *Then*

$$\sum_{i=1}^{k+2} d_{P \cup L_j}(x_i) \leq \begin{cases} |P \cup L_j| + k + 1, v_p \in \cap_{i=1}^{k+2} N_P(x_i), \\ |P \cup L_j| + k, \qquad v_p \notin \cap_{i=1}^{k+2} N_P(x_i). \end{cases}$$

Proof. If $L^*_j = \emptyset$, then for each pair x_i and x_j with $i < j$, $N^{-s}_P(x_i) \cap N_P(x_j) = \emptyset$, $s \geq 1$. The result holds. So assume that $|L^*_j| \geq k+1$. We prove the Lemma by induction on $|P|$. If $|P| = 1, 2$, the result is trivial. If $|P| = 3$, $L^*_j = R^*(v_2)$. Since $|L^*_j| \geq k+1$, $|P \cup L_j| \geq k+3$. Then

$$\begin{aligned}\sum_{i=1}^{k+2} d_{P\cup L_j}(x_i) &\le k+2+\begin{cases} k+2, v_p \in \cap_{i=1}^{k+2} N_P(x_i),\\ k+1, v_p \notin \cap_{i=1}^{k+2} N_P(x_i),\end{cases}\\ &= k+3+\begin{cases} k+1, v_p \in \cap_{i=1}^{k+2} N_P(x_i),\\ k, \quad v_p \notin \cap_{i=1}^{k+2} N_P(x_i),\end{cases}\\ &\le \begin{cases} |P\cup L_j|+k+1, v_p \in \cap_{i=1}^{k+2} N_P(x_i),\\ |P\cup L_j|+k, \quad v_p \notin \cap_{i=1}^{k+2} N_P(x_i).\end{cases}\end{aligned}$$

Now assume the result holds for path $|P'| < |P|$. Suppose that x_q and x_p ($q < p$) is the first pair such that $N_P^{-s}(x_q) = N_P(x_p)$, $s \ge 2$, and $N_P^{-j}(x_q) \cap (\cup_{i=1}^{k+2} N_P(x_i)) = \emptyset$, $1 \le j \le s-1$. Denote $N_P(x_p) = y_p$, $N_P(x_q) = y_q$. Take $P_1 = P[v_1, y_p]$, $P_2 = P[y_p^+, y_q^-]$ and $P_3 = P[y_q, v_p]$. Then $P_1 \cup L_j^1 = P_1$, $P_2 \cup L_j^2 = L_{j_{pq}}^h$ and $P_3 \cup L_j^3 = P_3 \cup L_j^* - L_{j_{pq}}^h$. By claim 2, $|L_{j_{pq}}^h| \ge k+1$. By induction hypothesis, it holds that

$$\sum_{i=1}^{k+2} d_{P_1\cup L_j^1}(x_i) \le |P_1 \cup L_j^1| + k + 1$$

and

$$\sum_{i=1}^{k+2} d_{P_3\cup L_j^3}(x_i) \le \begin{cases} |P_3\cup L_j^3|+k+1, v_p \in \cap_{i=1}^{k+2} N_P(x_i),\\ |P_3\cup L_j^3|+k, \quad v_p \notin \cap_{i=1}^{k+2} N_P(x_i).\end{cases}$$

Then

$$\begin{aligned}\sum_{i=1}^{k+2} d_{P\cup L_j}(x_i) &\le |P_1\cup L_j^1|+k+1+\begin{cases} |P_3\cup L_j^3|+k+1, v_p \in \cap_{i=1}^{k+2} N_P(x_i),\\ |P_3\cup L_j^3|+k, \quad v_p \notin \cap_{i=1}^{k+2} N_P(x_i),\end{cases}\\ &\le |P_1\cup L_j^1|+|P_2\cup L_j^2|+|P_3\cup L_j^3|+\begin{cases} k+1, v_p \in \cap_{i=1}^{k+2} N_P(x_i),\\ k, \quad v_p \notin \cap_{i=1}^{k+2} N_P(x_i),\end{cases}\\ &= \begin{cases} |P\cup L_j|+k+1, v_p \in \cap_{i=1}^{k+2} N_P(x_i),\\ |P\cup L_j|+k, \quad v_p \notin \cap_{i=1}^{k+2} N_P(x_i).\end{cases}\end{aligned}$$

The result holds. □

For any distinguish vertices $y_0, y_1, ..., y_p$, we define $\varphi(y_0|y_1, ..., y_p) = 1$ if $y_0 \in \cap_{i=1}^{p} N(y_i)$ and $\varphi(y_0|y_1, ..., y_p) = 0$ if $y_0 \notin \cap_{i=1}^{p} N(y_i)$. For $1 \le i \le k+2$, by Lemma 2.2, we have

$$\begin{aligned}\sum_{j=1}^{k+2} d_{C(v_i,v_{i+1}]\cup L_i}(x_j) &\le |C(v_i, x_i)| + |C(x_i, v_{i+1}] \cup L_i| + k + \varphi(v_{i+1}|x_1, ..., x_{k+2})\\ &= |C(v_i, v_{i+1}] \cup L_i| + k - 1 + \varphi(v_{i+1}|x_1, ..., x_{k+2}).\end{aligned}$$

For $i > k+2$, by Lemma 2.2 again, we have

$$\sum_{j=1}^{k+2} d_{C(v_i,v_{i+1}]\cup L_i}(x_j) \le |C(w_i, v_{i+1}) \cup L_i| + k + \varphi(v_{i+1}|x_1, ..., x_{k+2}).$$

By the definition of $x_i (i = 1, 2, ..., k+2)$, L_i and Lemma 2.1, $x_1, x_2, ..., x_{k+2}$ have no neighbor in $H \cup (\bigcup_{j=k+3}^{t} R(v_j, w_j))$ and any pair of $x_1, x_2, ..., x_{k+2}$ have no common neighbor in $G - C \cup (\bigcup_{i=1}^{t} L_i)$. Hence

$$\sum_{i=1}^{k+2} d_{G-C\cup(\bigcup_{i=1}^{t} L_i)}(x_i) \le |G| - |C| - |\bigcup_{i=1}^{t} L_i| - |H| - |\bigcup_{i=k+3}^{t} R(v_i, w_i)|.$$

Thus

$$\begin{aligned}
&\sum_{i=0}^{k+2} d(x_i) \\
\le\ & |H| - 1 + t + \sum_{i=1}^{k+2}(|C(v_i, v_{i+1}] \cup L_i| + k - 1 + \varphi(v_{i+1}|x_1, ..., x_{k+2})) \\
& + \sum_{i=k+3}^{t} (|C(w_i, v_{i+1}) \cup L_i| + k + \varphi(v_i|x_1, ..., x_{k+2})) + |G| - |H| - |C| \\
& - \sum_{i=1}^{t} |L_i| - \sum_{i=k+3}^{t} |R(C(v_i, w_i))| \\
=\ & n - 1 + t + \sum_{i=1}^{k+2}(|C(v_i, v_{i+1}]| + k - 1) + \sum_{i=k+3}^{t} (|C(w_i, v_{i+1})| + k) \\
& - \sum_{i=k+3}^{t} |R(C(v_i, w_i))| - |C| + \sum_{i=1}^{t} \varphi(v_i|x_0, x_1, ..., x_{k+2}) \\
=\ & n - 1 + t + (k+2)(k-1) + k(t-k-2) - \sum_{i=k+3}^{t} |R^*(C(v_i, w_i))| + |\cap_{i=0}^{k+2} N(x_i)| \\
\le\ & n - 1 + t + (k+2)(k-1) + k(t-k-2) - (k+1)(t-k-2) + |\cap_{i=0}^{k+2} N(x_i)| \\
=\ & n - 1 + k(k+2) + |\cap_{i=0}^{k+2} N(x_i)|.
\end{aligned}$$

That is, $\bar{\sigma}_{k+3}(G) \le n - 1 + k(k+2)$. This contradiction concludes the proof of Theorem 1.8. □

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